# Triangular Arbitrage in the Foreign Exchange Market

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**Summary.** In the present article, we review two of our previous works. First, we show that there are in fact triangular arbitrage opportunities in the spot foreign exchange markets, analyzing the time dependence of the yen-dollar rate, the dollar-euro rate and the yen-euro rate. Second, we propose a model of foreign exchange rates with an interaction. The model includes effects of triangular arbitrage transactions as an interaction among three rates. The model explains the actual data of the multiple foreign exchange rates well. Finally, we suggest, on the basis of the model, that triangular arbitrage makes the auto-correlation function of foreign exchange rates negative in a short time scale.

**Keywords.** Foreign exchange, Triangular arbitrage, Auto correlation, Stochastic model

## 1 Introduction

We recently pointed out the existence of the triangular arbitrage opportunity in the foreign exchange market [1, 2]. The triangular arbitrage is a financial activity that takes advantage of the three foreign exchange rates among three currencies [3]. It makes the product of the three foreign exchange rates converge to its average, thereby generating an interaction among the rates.

In order to study effects of the triangular arbitrage on the fluctuations of the exchange rates, we introduced [1] a stochastic model describing the time evolution of the exchange rates with an interaction. The model successfully described the fluctuation of the data of the real market.

We showed further [2] that our model gives an explanation to an interesting feature of the fluctuation of foreign exchange rates. The auto-correlation function of the fluctuation of the foreign exchange rates has been known to be negative in a short time scale [4]. Our model suggests that an important ingredient of the negative auto-correlation is the triangular arbitrage.

# 2 The triangular arbitrage as an interaction

The triangular arbitrage is a financial activity that takes advantage of three exchange rates. When a trader exchanges one Japanese yen to some amount of US dollar, exchanges the amount of US dollar to some amount of euro and exchanges the amount of euro back to Japanese yen instantly at time $t$, the final amount of Japanese yen is given by

$$\mu \equiv \prod_{i=1}^{3} r_i(t), \tag{1}$$

where

$$r_1(t) \equiv \frac{1}{\text{yen-dollar ask } (t)} \tag{2}$$

$$r_2(t) \equiv \frac{1}{\text{dollar-euro ask } (t)} \tag{3}$$

$$r_3(t) \equiv \text{yen-euro bid } (t). \tag{4}$$

If the rate product $\mu$ is greater than unity, the trader can make profit through the above transaction. This is the triangular arbitrage transaction. Once there is a triangular arbitrage opportunity, many traders will make the transaction. This makes $\mu$ converge to a value less than unity, thereby eliminating the opportunity. Triangular arbitrage opportunities nevertheless appear, because each rate $r_i$ fluctuates strongly.

The probability density function of the rate product $\mu$ (Fig. 1) has a sharp peak and fat tails. It means that the fluctuations of the exchange rates have correlation that makes the rate product converge to its average $\langle \mu \rangle \simeq 0.99998$. The average is less than unity because of the spread; the spread is the difference between the ask and the bid prices and is usually of the order of 0.05% of the prices.

For later convenience, we here define the logarithm rate product $\nu$ as the logarithm of the product of the three rates:

$$\nu(t) = \ln \prod_{i=1}^{3} r_i(t) = \sum_{i=1}^{3} \ln r_i(t). \tag{5}$$

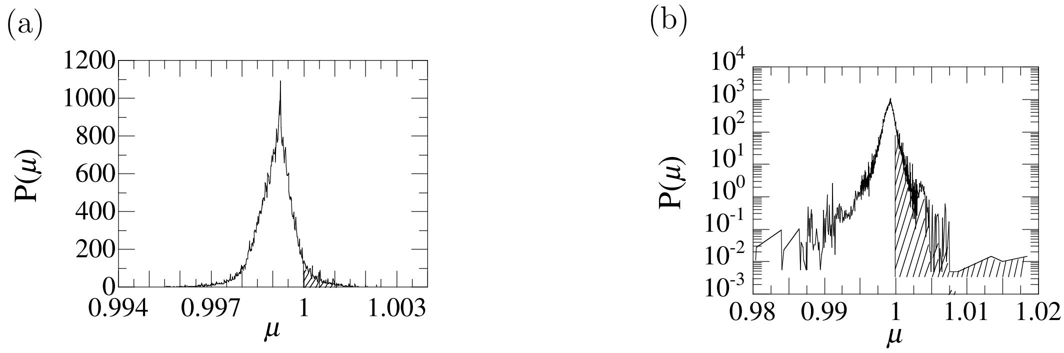


Figure 1: The probability density function of the rate product $\mu$. (b) is a semi-logarithmic plot of (a). The shaded area represents triangular arbitrage opportunities. The data were taken from January 25 1999 to March 12 1999.

There is a triangular arbitrage opportunity whenever this value is positive.

In one of our previous works [1], we constructed a stochastic model of the time evolution of foreign exchange rates that takes account of the effect of the triangular arbitrage transaction. The basic equation of this model is the time evolution of the logarithm of each rate:

$$\ln r_i(t+T) = \ln r_i(t) + f_i(t) + g(\nu(t)), \quad (i = 1, 2, 3) \tag{6}$$

where $f_i$ denotes independent fluctuation that obeys a truncated Lévy distribution [5] and $g$ represents an interaction function defined by

$$g(\nu) = -a(\nu - \langle\nu\rangle), \tag{7}$$

where $a$ is a positive constant which specifies the interaction strength and $\langle\nu\rangle$ is the time average of $\nu$. The time-evolution equation of the logarithm rate product $\nu$ is given by summing eq. (6) over all $i$:

$$\nu(t+T) - \langle\nu\rangle = (1 - 3a)(\nu(t) - \langle\nu\rangle) + \sum_{i=1}^{3} f_i(t). \tag{8}$$

The model equation (8) well describes a fat-tail probability distribution of $\nu(t)$ of the actual market (Fig. 2) [1].

From the physical viewpoint, we can regard the model equation (6) as a one-dimensional random walk of three particles with a restoring force, by making $\ln r_i$ the position of each particle (Fig. 3). The logarithm rate product $\nu$ is the summation of $\ln r_i$, hence is proportional to the center of gravity of the three particles. The restoring force $g(\nu)$ makes the center of gravity converge to a certain point $\langle\nu\rangle$. The form of the restoring force (7) is the same as that of the harmonic oscillator.

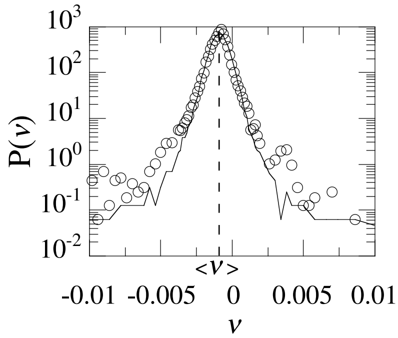


Figure 2: The probability density function of $\nu$. The circle (∘) denotes the real data and the solid line denotes our simulation data. The simulation data fit the real data well.

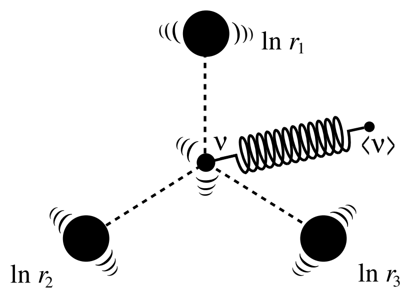


Figure 3: A schematic image of the model. The three random walker with the restoring force working the center of gravity.

# 3 Negative auto-correlation of the foreign exchange rates in a short time scale

In the other of our previous works [2], we pointed out another consequence of the triangular arbitrage, namely the negative auto-correlation of each exchange rate in a short time scale. Let us first show it in the actual data. We analyzed actual tick-by-tick data of the yen-dollar rate, the dollar-euro rate and the yen-euro rate, taken from January 25, 1999 to March 12, 1999 except for the weekends.

The auto-correlation function of the rate fluctuation is defined by the following formula:

$$c_i(n) = \frac{\langle \Delta r_i(t+nT)\Delta r_i(t)\rangle - \langle \Delta r_i(t)\rangle^2}{\langle \Delta r_i(t)^2\rangle - \langle \Delta r_i(t)\rangle^2}, \quad (i = 1,2,3; n = 0,1,2,...), \tag{9}$$

where

$$\Delta r_i(t) \equiv \ln \frac{r_i(t+T)}{r_i(t)} \ (i = 1,2,3), \tag{10}$$

and the angular brackets $\langle ... \rangle$ denote the time average. We fixed the time step $T$ at one minute.

Figure 4 shows that the auto-correlation function of each rate has a negative value for $n = 1$. We here claim that the triangular arbitrage is one of the major causes of this negative auto-correlation. In order to see it, we simulated eq. (6) and calculated the auto-correlation function (9). The simulation data (also shown in Fig. 4) are qualitatively consistent with the behavior of the auto-correlation function of the actual data.

Another analysis is possible. Using eq. (6), we can rewrite the auto-correlation function

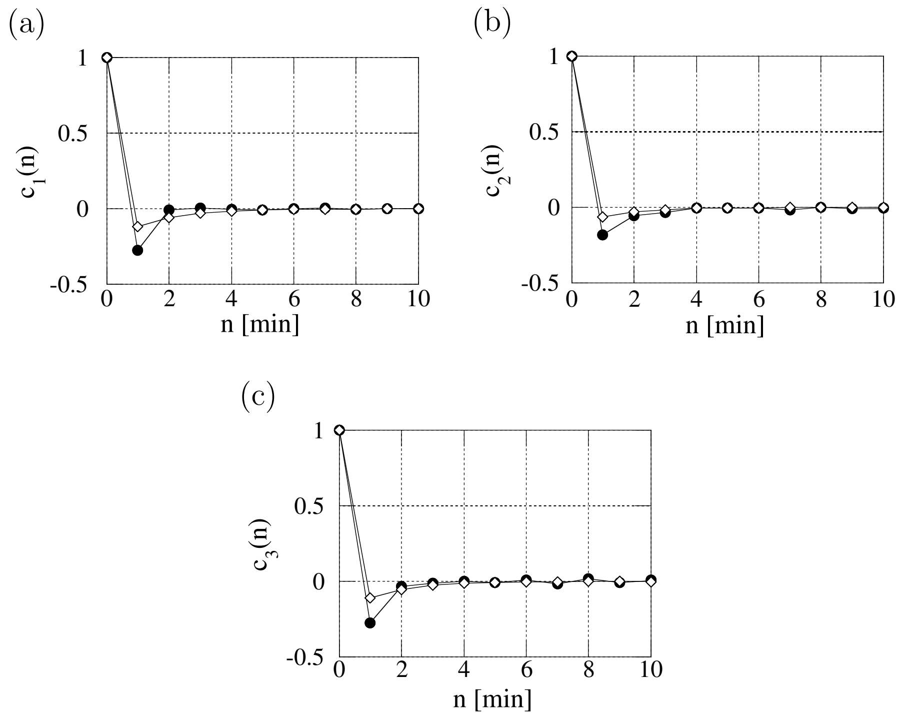


Figure 4: The auto-correlation function of the rate change of the actual data: (a) $c_1(n)$; (b) $c_2(n)$; (c) $c_3(n)$. The circles ($\bullet$) denote the actual data and the diamonds ($\diamond$) denote the simulation data.

Table 1: The value of $c_i(n=1)$ from the actual data, the simulation data and eq. (12).

| Rate | Actual Data | Simulation | Eq. (12) |
|---|---|---|---|
| $r_1$ (1/yen-dollar ask) | $-0.27$ | $-0.12$ | $-0.12$ |
| $r_2$ (1/dollar-euro ask) | $-0.18$ | $-0.061$ | $-0.095$ |
| $r_3$ (yen-euro bid) | $-0.28$ | $-0.11$ | $-0.13$ |

(9) for $n = 1$ as:

$$c_i(n=1) = \frac{\langle (f_i(t+T)+g(t+T))(f_i(t)+g(t))\rangle - \langle f_i(t)+g(t)\rangle^2}{\langle (f_i(t)+g(t))^2\rangle - \langle f_i(t)+g(t)\rangle^2} \tag{11}$$

$$= -a\frac{\sigma_{f_i}^2 - a(1-3a)\sigma_\nu^2}{\sigma_{f_i}^2 + a^2\sigma_\nu^2}, \tag{12}$$

where ${\sigma_x}^2$ denotes the variance of the variable $x$. We here used the following relations:

$$\langle f_i(t)\rangle = 0 \quad \text{and} \quad \langle f_i(t+T)f_i(t)\rangle = 0. \tag{13}$$

Note that we have $c_i(n=1) \approx -a < 0$ for small $a$.

We can estimate $\sigma_\nu$ and $\sigma_{f_i}$ from the market data. The auto-correlation function for $n = 1$ thus-estimated is compared in Table 1 to the one from the actual data and the one from the simulation data.

The value of $c_i(n=1)$ from the actual data is less than those from the simulation data and eq. (12). This may suggest that there are contributions from the triangular arbitrage of other combinations of three rates; for example, the triangular arbitrage among Japanese yen, US dollar and British pound.

## 4 Conclusions

We first showed that triangular arbitrage opportunities exist in the foreign exchange market. The rate product $\mu$ fluctuates around its average. Next, we introduced a model including the interaction caused by the triangular arbitrage transaction. Finally, on the basis of the model, we showed that the triangular arbitrage makes the auto-correlation function of each rate negative for $n = 1$. The comparison with the actual data is good qualitatively, but it also suggests that the triangular arbitrage of various combinations must be considered.

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