Taking the Arbitrage Strategy to More Than Three (N) Currencies

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Abstract

Until recently, the pervasive practice among multinational banks was to quote all currencies against the U.S. dollar for trading in the foreign exchange market. Now, however, a growing percentage of currency trades do not involve the dollar. For example, German, British, Japanese, and Swiss banks may quote the euro in terms of the pound sterling, the Japanese yen or the Swiss franc. Currency traders are continually alert to the possibility of taking advantage of exchange rate inconsistencies in different money centers, through arbitrage transactions. A number of international finance literatures explain the currency arbitrage process via a "triangular arbitrage" framework, assuming a single quote per bank, i.e., assuming no bid and asked spread or no transaction cost. Such approach typically compares market rates with inter-bank cross rates to determine when arbitrage is possible, and under such a simplifying assumption, the arbitrage condition is rather simple. In the real world, however, banks always use bid and asked prices, and they announce their quotations for <u>more than</u> three currencies (i.e., for "n" currencies), and this makes the explanation of the arbitrage conditions rather complicated. The purpose of this paper is to provide a rigorous derivation and explanation of n-currency arbitrage conditions with bid and asked prices as well as the implications to arbitrage strategy. The authors have not found such a derivation or explanation in any academic journal or textbook.

SECTION I: INTRODUCTION

Until recently, the pervasive practice among bank dealers was to quote all currencies against the U.S. dollar for trading among them. Now, however, a growing percentage of currency trades do not involve the dollar [Shapiro, 2003]. For example, German, British, or Japanese banks may quote the euro in terms of the pound sterling, the Japanese yen, or the Swiss franc. Currency traders are continually alert to the possibility of taking advantage of exchange rate inconsistencies in different money centers, through arbitrage transactions.

A number of international finance literatures explain the currency arbitrage process via a "triangular arbitrage" framework, assuming a single quote per bank, i.e., assuming no bid and asked spread or no transaction cost.[1] [Eiteman, 2004; Levtich, 1998; Madura, 2003, Shapiro, 2003; Eaker, Fabozzi and Grant, 1996; Grabbe, 1991; Kim and Kim, 1999]. Such approach typically compares market rates with inter-bank cross rates to determine when arbitrage (producing a positive profit) is possible, and under such a simplifying assumption, the arbitrage condition is rather simple. In the real world, however, banks always use bid and asked prices[2] when they announce their quotations for "n" currencies and this makes the explanation or derivation of the arbitrage conditions rather complicated. The purpose of this paper is to provide a rigorous derivation of n-currency arbitrage conditions with bid and asked prices as well as the implications to arbitrage strategy. The authors have not found such a derivation in any academic journal or textbook.

In Section II, we discuss the triangular arbitrage with no transaction cost, which is followed by triangular conditions based on bid and asked prices in Section III. In Sections IV and V, we extend the arbitrage condition to n currencies, i.e., n-currency case with no transaction cost in Section IV and n-currency case with bid and asked prices in Section V, respectively. Section VI provides concluding comments.

SECTION II: ARBITRAGE CONDITION WITH NO TRANSACTION COST

In order to facilitate the exposition, we start out with simple numerical examples which ignore bid and asked prices (i.e., assume zero transaction cost) and assume that a fourth party trader can either buy or sell currencies at the market rates quoted below (see the second columns of Tables 1 and 2) by three banks, i.e., Bank A in New York, Bank B in London, and Bank C in Frankfurt, respectively. (These are typical textbook examples. The market rate may be a mid-point between bid and asked prices.)

Table 1. Market Rates and Cross Rates between Banks: Numerical Example 1

	Market Rates	Cross Rates between Banks	Overvaluation or Undervaluation
Bank A quotes	\$1.5422/£	\$1.5402/£	£ is overvalued; \$ is undervalued.
Bank B quotes	£0.6006/€	£0.5998/€	€ is overvalued; £ is undervalued.
Bank C quotes	€1.0810/\$	€1.0796/\$	\$ is overvalued; € is undervalued.

You are a fourth party, an American trader. If you start out with \$1, how much triangular arbitrage profit can you make? A useful first step to answer the question may be to derive cross rates between banks. For example, one can derive the cross rate for Bank A by inverting the product of the quotes of the other two banks (Bank B and Bank C) -- note the symmetry of units in quotations. The second step is to determine, using the cross rate as a reference, which currency is overvalued or undervalued. In order to make an arbitrage profit, all you have to do is to “buy low,” i.e., buy a currency undervalued in terms of another currency, and “sell high,” i.e., sell a currency overvalued in terms of another currency.

Step 1: At Bank C, sell \$1 and buy €1.0810.

Step 2: At Bank B, sell €1.0810 and buy £0.6006 x 1.0810.

Step 3: At Bank A, sell £0.6006 x 1.0810 and buy \$1.5422 x 0.6006 x 1.0810 = \$1.001271.

You started with \$1 (or \$1,000,000) and ended up with \$1.001271 (or \$1,001,271), making a triangular arbitrage profit of \$0.001271 (or \$1,271).

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Table 2. Market Rates and Cross Rates between Banks: Numerical Example 2

	Market Rates	Cross Rates between Banks	Overvaluation or Undervaluation
Bank A quotes	\$1.5402/£	\$1.5725/£	£ is undervalued; \$ is overvalued.
Bank B quotes	£0.5885/€	£0.6008/€	€ is undervalued; £ is overvalued.
Bank C quotes	€1.0806/\$	€1.1033/\$	\$ is undervalued; € is overvalued.

Table 2 displays different market rates under which the currencies deemed “overvalued” (“undervalued”) in Table 1 are now “undervalued” (“overvalued”). For example, at Bank A, the pound sterling is undervalued, and the U.S. dollar overvalued. Again, if you start out with \$1, how much triangular arbitrage profit can you make?

Step 1: At Bank A, sell \$1 and buy £$\frac{1}{1.5402}$.

Step 2: At Bank B, sell £$\frac{1}{1.5402}$ and buy €$\frac{1}{1.5402}\frac{1}{0.5885}$.

Step 3: At Bank C, sell €$\frac{1}{1.5402}\frac{1}{0.5885}$ and buy \$$\frac{1}{1.5402}\frac{1}{0.5885}\frac{1}{1.0806}$ = \$$\frac{1}{0.97946416}$

= \$1.020966.

You started with \$1 (or \$1,000,000) and ended up with \$1.020966 (or \$1,020,966), making a triangular arbitrage profit of \$0.020966 (or a whopping \$20,966).

Using general symbols, the above process can be explained as follows:

Table 3. Market Rates and Cross Rates Using General Symbols

	Market Rates	Cross Rates between Banks
Bank A quotes	\$*A*/£	$\$\frac{1}{BC}/£$
Bank B quotes	£*B*/€	$£\frac{1}{CA}/€$
Bank C quotes	€*C*/\$	$€\frac{1}{AB}/\$$

Note that \$*A*/£ denotes Bank A's price of the pound sterling in terms of the U.S. dollar, etc.

Step 1: At Bank C, sell \$1 and buy €*C*.

Step 2: At Bank B, sell €*C* and buy £*BC*.

Step 3: At Bank A, sell £*BC* and buy \$*ABC*.

You started with \$1 and ended up with \$*ABC*. So, as long as $ABC > 1$, you can make an arbitrage profit. (If $ABC = 1$, no arbitrage opportunity or zero profit.) Note that, due to the symmetry of the units for market rates, given the market rate of one bank for one currency, the corresponding cross rate of the same currency is the reciprocal of the product of the other two bank's market rates (for example, A vs. $1/BC$, B vs. $1/CA$). Therefore, ABC is the ratio of the market rate of one currency to its cross rate, and this property remains valid regardless of the currency and bank. This implies that $ABC > 1$ implies that the market rates are greater than the cross rates for all three banks and for all three currencies.

Alternatively, you can do transactions in the following order:

Step 1: At Bank A, sell \$1 and buy £$\frac{1}{A}$.

Step 2: At Bank B, sell £$\frac{1}{A}$ and buy €$\frac{1}{AB}$.

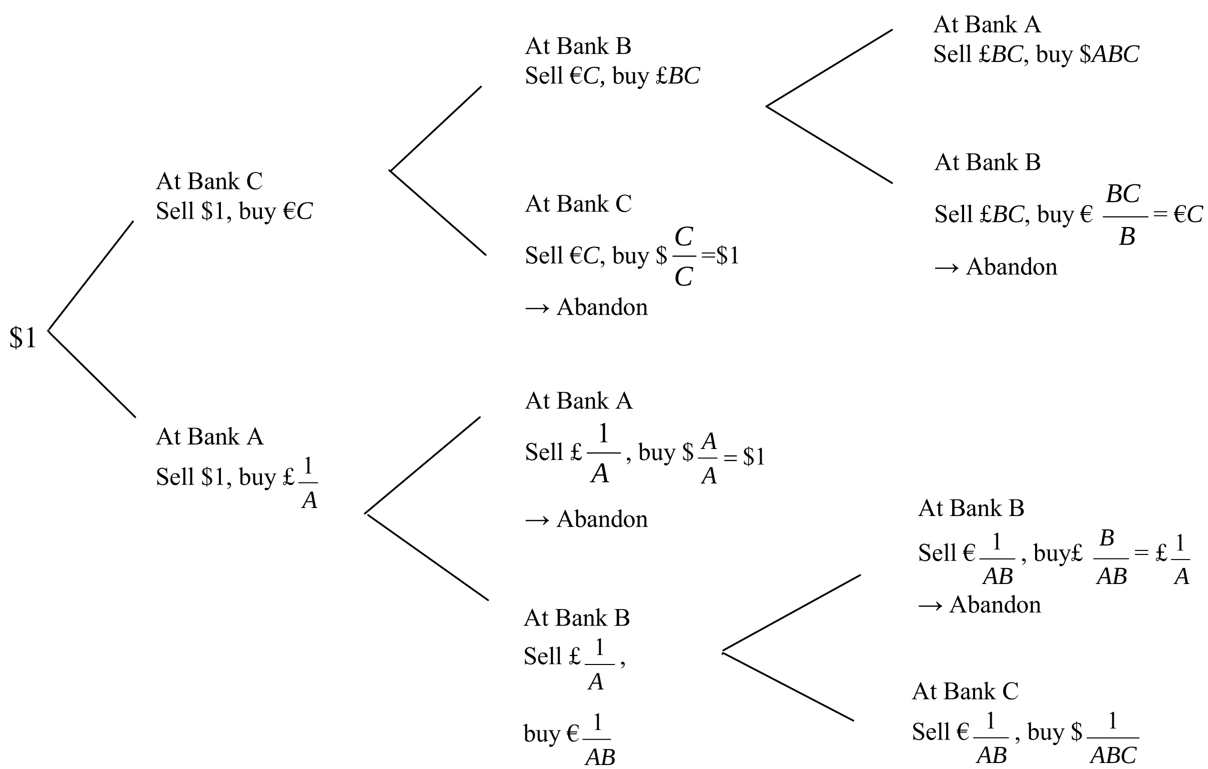
Step 3: At Bank C, sell €$\frac{1}{AB}$ and buy \$$\frac{1}{ABC}$.

As long as \$$\frac{1}{ABC}$>\$1, or $ABC < 1$, you can make an arbitrage profit. Due to the symmetry of the units for market rates, $ABC < 1$ implies that the market rates are smaller than the cross rates for all three banks and for all three currencies.

Employing a decision tree, the above steps can be explained in a compact manner as shown by Figure 1.

Figure 1. A Decision Tree for Determining Arbitrage Conditions: With Zero Transaction Cost

Step 1 Step 2 Step3



SECTION III: ARBITRAGE CONDITIONS BASED ON BID AND ASKED PRICES

Real world market rates (bid and asked prices) quoted by banks and corresponding cross rates between banks can be expressed, using general symbols, as shown by Table 4 below.

Table 4. Market Rates and Cross Rates in terms of Bid and Asked Prices

	Market Rates		Cross Rates between Banks	
	Bid	Asked	Bid	Asked
Bank A quotes	$$A_b$/£	$$A_a$/£	$$\frac{1}{B_aC_a}$/£	$$\frac{1}{B_bC_b}$/£
Bank B quotes	£B_b/€	£B_a/€	£$\frac{1}{C_aA_a}$/€	£$\frac{1}{C_bA_b}$/€
Bank C quotes	€C_b/$	€C_a/$	€$\frac{1}{A_aB_a}$ /$	€$\frac{1}{A_bB_b}$ /$

Note that $\$A_b/£$ denotes Bank A's bid price of the pound sterling in terms of the U.S. dollar, while $\$A_a/£$ means Bank A's asked price of the pound sterling in terms of the U.S. dollar, etc.

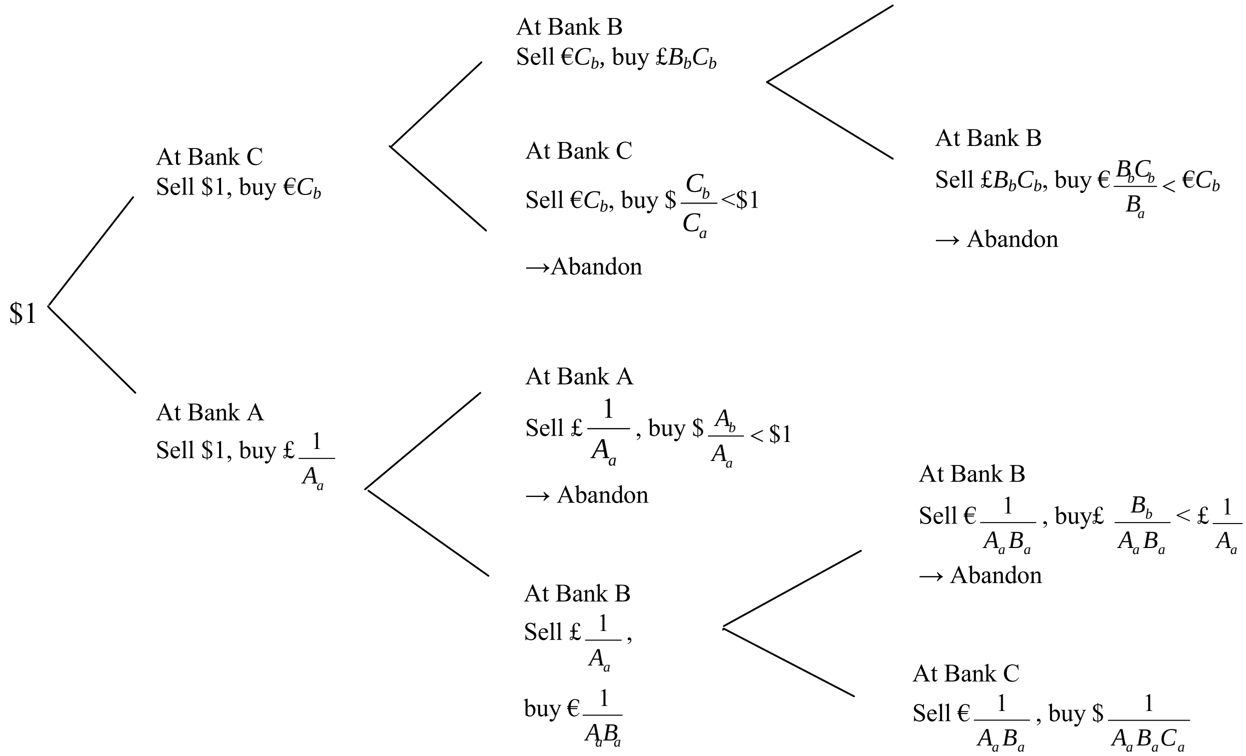
Claim: There is an arbitrage profit opportunity if $A_bB_bC_b > 1$ or $A_aB_aC_a < 1$. Equivalently, there is no arbitrage opportunity if $A_bB_bC_b \leq 1 \leq A_aB_aC_a$.

Note: Because of the relation between the bid and asked prices, $A_b < A_a$, $B_b < B_a$, and $C_b < C_a$, either $A_aB_aC_a > A_bB_bC_b \geq 1$ or $A_bB_bC_b < A_aB_aC_a \leq 1$. $A_bB_bC_b = A_aB_aC_a = 1$ implies there is neither loss nor gain.

If you start out with \$1, there are two choices in Step 1. You either sell the \$1 and buy €C_b at Bank C or sell the \$1 and buy £ $\frac{1}{A_a}$ at Bank A. Steps 2 and 3 are explained in a similar manner, employing a decision tree (see Figure 2 below). When you make two consecutive transactions at the same bank, (e.g. in Step 2 at Bank C, sell €C_b and buy \$ $\frac{C_b}{C_a}$ at Bank C), you end up with less fund than you started at that bank (no wonder, for you have to pay the transaction cost), you should abandon the project.

Figure 2. A Decision Tree Determining Triangular Arbitrage Conditions: With Bid and Asked Prices

Step 1 | Step 2 | Step3



You end up with \$$A_bB_bC_b$ if you sell the initial \$1 at Bank C, or with \$$\frac{1}{A_aB_aC_a}$ if you sell it at Bank A. Thus, in order to make a profit, you should end up with either \$$A_bB_bC_b$ > \$1 (i.e., $A_bB_bC_b > 1$) or \$$\frac{1}{A_aB_aC_a}$ > \$1 (i.e., $A_aB_aC_a < 1$). Hence, you should do the following: If $A_bB_bC_b > 1$, do transactions at Bank C, B, A order (upper branch), whereas if $A_aB_aC_a < 1$, do transactions at Bank A, B, C order (lower branch).

Note that $A_bB_bC_b > 1$ implies $A_b > \frac{1}{B_bC_b}$, $B_b > \frac{1}{C_bA_b}$, $C_b > \frac{1}{A_bB_b}$, which, in turn, implies that the bid market rates are greater than the asked cross rates for all three banks and for all three currencies. The converse is obvious. Therefore, we conclude that there is an arbitrage opportunity if and only if the bid market rates are greater than the asked cross rates for all three currencies and for all three banks.

Similarly, $A_aB_aC_a < 1$ implies $A_a < \frac{1}{B_aC_a}$, $B_a < \frac{1}{C_aA_a}$, $C_a < \frac{1}{A_aB_a}$ and vice versa, which again implies that there is an arbitrage opportunity if and only if the asked market rates are smaller than the bid cross rates for all three currencies and for all three banks.

By way of numerical examples shown by Tables 5, 6 and 7 below, the arbitrage conditions and the three steps to be taken by the arbitrager may be explained more easily.

Table 5. Numerical Example I for Bid and Asked Rates

	Market Rates		Cross Rates (between banks)	
	Bid	Asked	Bid	Asked
Bank A's quotes	\$1.5417/£	\$1.5427/£	\$1.5390/£	\$1.5415/£
Bank B's quotes	£0.6004/€	£0.6008/€	£0.5994/€	£0.6003/€
Bank C's quotes	€1.0805/\$	€1.0815/\$	€1.0789/\$	€1.0803/\$

Since $A_bB_bC_b$ = 1.5417 x 0.6004 x 1.0805 = 1.000150 > 1 – implying that all bid market rates are greater than all asked cross rates –, you follow the upper branch of the decision tree, i.e., sell the (overvalued) dollar for the (undervalued) euro at Bank C, sell the euro for the pound sterling at Bank B, and sell the pound sterling for the dollar at Bank A, ending with a profit of \$0.00015 (\$150) if you started with \$1 (\$1,000,000).

Table 6. Numerical Example II for Bid and Asked Rates

	Market Rates		Cross Rates (between banks)	
	Bid	Asked	Bid	Asked
Bank A's quotes	\$1.5397/£	\$1.5407/£	\$1.5712/£	\$1.5738/£
Bank B's quotes	£0.5883/€	£0.5887/€	£0.6004/€	£0.6013/€
Bank C's quotes	€1.0801/\$	€1.0811/\$	€1.1025/\$	€1.1040/\$

Since $A_aB_aC_a$ = 1.5407 x 0.5887 x 1.0811 = 0.980569 < 1 – implying that all asked market rates are smaller than all bid cross rates –, you follow the lower branch of the decision tree, i.e., sell the (overvalued) dollar for the (undervalued) pound sterling at Bank A, sell the pound sterling for the euro at

Bank B, and sell the euro for the dollar at Bank C, ending with a profit of \$0.019816 (\$19,816) if you started with \$1 (\$1,000,000).

Table 7. Numerical Example III for Bid and Asked Rates

	Market Rates		Cross Rates (between banks)	
	Bid	Asked	Bid	Asked
Bank A's quotes	\$1.5402/£	\$1.5442/£	\$1.5351/£	\$1.5454/£
Bank B's quotes	£0.5997/€	£0.6015/€	£0.5980/€	£0.6017/€
Bank C's quotes	€1.0790/\$	€1.0830/\$	€1.0766/\$	€1.0827/\$

Since $A_bB_bC_b$ = 1.5402 x 0.5997 x 1.0790 = 0.9966 < 1, no arbitrage profit can be made. Alternatively, since $A_aB_aC_a$ = 1.5442 x 0.6015 x 1.0830 = 1.0059 > 1, no arbitrage profit can be made, either. Note that the quotations used in Table 1 above are mid-points of the bid and asked prices in Table 7. Using the quotes of Table 1, you can make a triangular arbitrage profit. Given the bid and asked quotes of Table 7, however, the transaction cost (imposed by the spread between bid and asked prices) wipes out any potential profit that may be realized in the zero-transaction-cost world. In fact, the spreads between bid and asked prices (transaction costs) imposed by the three banks are so wide that you would end up with a loss regardless of which branch of the decision tree you followed.

SECTION IV: N-CURRENCY CASE WITH NO TRANSACTION COST

Let c_{ij} be the bid price of currency i in terms of currency j. This means by selling one unit of currency i one gets c_{ij} units of currency j. The unit of c_{ij}, is (unit of currency j/unit of currency i), and because there is no transaction cost, i.e., assuming bid price = asked price or no distinction of bid and the asked prices, $c_{ij} = \frac{1}{c_{ji}}$: by selling one unit of currency i, one gets c_{ij} units of currency j, which is equivalent to getting one unit of currency j by selling $\frac{1}{c_{ij}}$ units of currency i. Therefore we have the following bid-price matrix

$$\mathbf{C} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1n} \\ c_{21} & c_{22} & c_{23} & \cdots & c_{2n} \\ c_{31} & c_{32} & c_{33} & \cdots & c_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & c_{n3} & \cdots & c_{nn} \end{pmatrix} = \begin{pmatrix} 1 & c_{12} & c_{13} & \cdots & c_{1n} \\ \frac{1}{c_{12}} & 1 & c_{23} & \cdots & c_{2n} \\ \frac{1}{c_{13}} & \frac{1}{c_{23}} & 1 & \cdots & c_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{c_{1n}} & \frac{1}{c_{2n}} & \frac{1}{c_{3n}} & \cdots & 1 \end{pmatrix}$$

Theorem 1. There is an arbitrage profit opportunity if $c_{12}c_{23}c_{34}\cdots c_{(n-1)n}c_{n1} > 1$ or $c_{12}c_{23}c_{34}\cdots c_{(n-1)n}c_{n1} < 1$.

Proof. Start with one unit of currency 1: sell one unit of currency 1 and buy c_{12} units of currency 2. Sell this c_{12} units of currency 2 and buy $c_{12}c_{23}$ units of currency 3 (recall that one unit of currency 2 buys c_{23} units of currency 3). Continue until we have $c_{12}c_{23}c_{34}\cdots c_{(n-1)n}$ units of currency n. We sell this to get $c_{12}c_{23}c_{34}\cdots c_{(n-1)n}c_{n1}$ units of currency 1. We started with one unit of currency 1 and $c_{12}c_{23}c_{34}\ldots c_{(n-1)n}c_{n1} > 1$, hence we have gained $c_{12}c_{23}c_{34}\cdots c_{(n-1)n}c_{n1} - 1$ units in currency 1 by arbitrage.

Consider another case. We need to sell one unit of currency j to buy $\frac{1}{c_{ij}}$ units of currency i. Start with one unit of currency 1. Sell it to buy $\frac{1}{c_{n1}}$ units of currency n. Sell this $\frac{1}{c_{n1}}$ units of currency n to buy $\frac{1}{c_{(n-1)n}}\frac{1}{c_{n1}}$ units of currency $(n-1)$. Again sell this to buy $\frac{1}{c_{(n-2)(n-1)}}\frac{1}{c_{(n-1)n}}\cdot\frac{1}{c_{n1}}$ units of currency $(n-2)$. Continue selling currency i and buying currency $(i-1)$ until we have $\frac{1}{c_{12}}\cdot\frac{1}{c_{23}}\cdots\frac{1}{c_{(n-2)(n-1)}}\cdot\frac{1}{c_{(n-1)n}}\cdot\frac{1}{c_{n1}}$ units of currency 1. If $\frac{1}{c_{12}}\cdot\frac{1}{c_{23}}\cdots\frac{1}{c_{(n-2)(n-1)}}\cdot\frac{1}{c_{(n-1)n}}\cdot\frac{1}{c_{n1}} > 1$ or equivalently $c_{12}c_{23}c_{34}\cdots c_{(n-1)n}c_{n1} < 1$, then we have an arbitrage profit of $\frac{1}{c_{12}}\cdot\frac{1}{c_{23}}\cdots\frac{1}{c_{(n-2)(n-1)}}\cdot\frac{1}{c_{(n-1)n}}\frac{1}{c_{n1}} - 1$. □

Apply this result to the three currency case. Let the currency 1 be the U.S. dollar, the currency 2 be the British pound, and the currency 3 be the euro with the bid and asked prices between currencies. Then we have the following bid price matrix:

$$\mathbf{C} = \begin{pmatrix} 1 & \frac{1}{A} & C \\ A & 1 & \frac{1}{B} \\ \frac{1}{C} & B & 1 \end{pmatrix}$$

Then $c_{12}c_{23}c_{31} = \frac{1}{A}\cdot\frac{1}{B}\cdot\frac{1}{C} > 1$ implies $A\cdot B\cdot C < 1$ and $c_{12}c_{23}c_{31} = \frac{1}{A}\cdot\frac{1}{B}\cdot\frac{1}{C} < 1$ implies $A\cdot B\cdot C > 1$, as is expected from Theorem 1.

SECTION VI: N-CURRENCY CASE WITH BID AND ASKED PRICES

Let b_{ij} be the bid price of currency i in terms of currency j. This means by selling one unit of currency i one gets b_{ij} units of currency j.

Let a_{ij} be the asked price of currency i in terms of currency j. This means to buy one unit of currency i one should sell a_{ij} units of currency j. The unit of both prices, a_{ij} and b_{ij}, is (unit of currency j/unit of currency i), and a_{ij} is always greater than b_{ij}.

Note that $b_{ij} = \frac{1}{a_{ji}}$: by selling one unit of currency i, one gets b_{ij} units of currency j, which is equivalent to getting one unit of currency j by selling $\frac{1}{b_{ij}}$ units of currency i. By definition of the asked price a_{ji}, one should sell a_{ji} units of currency i to buy one unit of currency j. Therefore we have the following bid-price matrix

$$\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2n} \\ b_{31} & b_{32} & b_{33} & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & b_{nn} \end{pmatrix} = \begin{pmatrix} 1 & b_{12} & b_{13} & \cdots & b_{1n} \\ \frac{1}{a_{12}} & 1 & b_{23} & \cdots & b_{2n} \\ \frac{1}{a_{13}} & \frac{1}{a_{23}} & 1 & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{a_{1n}} & \frac{1}{a_{2n}} & \frac{1}{a_{3n}} & \cdots & 1 \end{pmatrix}$$

and asked-price matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} 1 & a_{12} & a_{13} & \cdots & a_{1n} \\ \frac{1}{b_{12}} & 1 & b_{23} & \cdots & a_{2n} \\ \frac{1}{b_{13}} & \frac{1}{b_{23}} & 1 & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{b_{1n}} & \frac{1}{b_{2n}} & \frac{1}{b_{3n}} & \cdots & 1 \end{pmatrix}.$$

Using the fact that $b_{ij} = \frac{1}{a_{ji}}$, we can get one matrix from the other by taking the reciprocal of each element and take the transpose.

Theorem 2. There is an arbitrage profit opportunity if $b_{12}b_{23}b_{34}\cdots b_{(n-1)n}b_{n1} > 1$ or $a_{12}a_{23}a_{34}\cdots a_{(n-1)n}a_{n1} < 1$. (Note that these two conditions are not equivalent. The first condition is equivalent to $a_{21}a_{32}a_{43}\cdots a_{n(n-1)}a_{1n} < 1$.)

Proo: There are two ways of building arbitrage strategies, one with bid prices and the other with asked prices. First consider the bid price case: start with one unit of currency 1: sell one unit of currency 1 and buy b_{12} units of currency 2. Sell this b_{12} units of currency 2 and buy $b_{12}b_{23}$ units of currency 3 (recall that one unit of currency 2 buys b_{23} units of currency 3). Continue to until we have $b_{12}b_{23}b_{34}\cdots b_{(n-1)n}$ units of currency n. We sell this to get $b_{12}b_{23}b_{34}\cdots b_{(n-1)n}b_{n1}$ units of currency 1. We started with one unit of currency 1 and $b_{12}b_{23}b_{34}\cdots b_{(n-1)n}b_{n1} > 1$, hence we have gained $b_{12}b_{23}b_{34}\cdots b_{(n-1)n}b_{n1} - 1$ units in currency

1 by arbitrage.

To use asked prices, recall that a_{ij} is the number of units of currency j to buy one unit of currency i. Therefore we need to sell one unit of currency j to buy $\frac{1}{a_{ij}}$ units of currency i. Start with one unit of currency 1. Sell it to buy $\frac{1}{a_{n1}}$ units of currency n. Sell this $\frac{1}{a_{n1}}$ units of currency n to buy $\frac{1}{a_{(n-1)n}}\frac{1}{a_{n1}}$ units of currency $(n-1)$. Again sell this to buy $\frac{1}{a_{(n-2)(n-1)}}\frac{1}{a_{(n-1)n}}\frac{1}{a_{n1}}$ units of currency $(n-2)$. Continue selling currency i and buying currency $(i-1)$ until we have $\frac{1}{a_{12}}\cdot\frac{1}{a_{23}}\cdots\frac{1}{a_{(n-2)(n-1)}}\cdot\frac{1}{a_{(n-1)n}}\cdot\frac{1}{a_{n1}}$ units of currency1. If $\frac{1}{a_{12}}\cdot\frac{1}{a_{23}}\cdots\frac{1}{a_{(n-2)(n-1)}}\cdot\frac{1}{a_{(n-1)n}}\cdot\frac{1}{a_{n1}}>1$ or equivalently $a_{12}a_{23}a_{34}\cdots a_{(n-1)n}a_{n1}<1$, then we have an arbitrage profit of $\frac{1}{a_{12}}\cdot\frac{1}{a_{23}}\cdots\frac{1}{a_{(n-2)(n-1)}}\cdot\frac{1}{a_{(n-1)n}}\frac{1}{a_{n1}}-1$. □

Apply this result to the three currency case. Let the currency 1 be the U.S. dollar, the currency 2 be the British pound, and the currency 3 be the euro with the bid and asked prices. Then we have the following bid price matrix:

$$\mathbf{B} = \begin{pmatrix} 1 & \frac{1}{A_a} & C_b \\ A_b & 1 & \frac{1}{B_a} \\ \frac{1}{C_a} & B_b & 1 \end{pmatrix} = \mathbf{A} = \begin{pmatrix} 1 & \frac{1}{A_b} & C_a \\ A_a & 1 & \frac{1}{B_b} \\ \frac{1}{C_b} & B_a & 1 \end{pmatrix}.$$

Then $b_{12}b_{23}b_{31}=\frac{1}{A_a}\cdot\frac{1}{B_a}\cdot\frac{1}{C_a}>1$ implies $A_a\cdot B_a\cdot C_a<1$ and $a_{12}a_{23}a_{31}=\frac{1}{A_b}\cdot\frac{1}{B_b}\cdot\frac{1}{C_b}<1$ implies $A_b\cdot B_b\cdot C_b>1$, as is expected from Theorem 2.

SECTION VI: CONCLUSION

In a world of zero transaction cost (typical textbook case), one can make an n-currency arbitrage profit if the product of the n market rates (assuming the rates are quoted "symmetrically") is either greater or smaller than 1. If such a product equals 1, the arbitrager ends up with a zero profit.

If foreign exchange market rates are quoted in terms of bid and asked prices (the real world case), an n-currency arbitrage profit can be made if (i) the product of the n bid market rates is greater than 1 (which, in turn, is equivalent to stating that all bid market rates are greater than all asked cross rates) or (ii) the product of the n asked market rates is smaller than 1 (which, in turn, is equivalent to stating that all asked market rates are smaller than all bid cross rates). Moreover, cases (i) and (ii) are mutually exclusive. If either the product of the n bid market rates or the product of the n asked market rates is equal to 1, the arbitrager ends up with a zero profit.

ENDNOTES

[1] Levi (1996), Eun and Resnick (2004), and Solnik and McLeavey utilize bid and asked prices when they discuss the triangular arbitrage transactions. However, unlike this paper, they focus on numerical examples without explicitly deriving arbitrage profit conditions.

[2] Dealers in financial markets including the foreign exchange market set their spreads between bid and asked prices to maximize their profits. Their spreads must be wide enough to allow them to recover their costs of doing business. Otherwise, they will not be profitable. Their spreads cannot be so wide, however, that no one will trade with them. For a detailed discussion on bid/asked spreads, see Harris (2003, pp. 297-321).

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